

# Travelling waves for a Frenkel-Kontorova chain

Boris Buffoni, Hartmut Schwetlick and Johannes Zimmer

July 2015

## Abstract

In this article, the Frenkel-Kontorova model for dislocation dynamics is considered, where the on-site potential consists of quadratic wells joined by small arcs, which can be spinodal (concave) as commonly assumed in physics. The existence of heteroclinic waves —making a transition from one well of the on-site potential to another— is proved by means of a Schauder fixed point argument. The setting developed here is general enough to treat such a Frenkel-Kontorova chain with smooth ( $C^2$ ) on-site potential. It is shown that the method can also establish the existence of two-transition waves for a piecewise quadratic on-site potential.

Mathematics Subject Classification: 37K60, 34C37, 58F03, 70H05

## 1 Introduction

In this article, we study the advance-delay difference-differential equation

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \psi'(u) = 0 \quad (1)$$

on  $\mathbb{R}$ , where  $\Delta_D$  is the discrete Laplacian,

$$\Delta_D u(x) := u(x+1) - 2u(x) + u(x-1);$$

the derivative  $g'(u)$  of the on-site potential

$$g(u) = \frac{1}{2} \alpha u^2 - \alpha \psi(u)$$

will be discussed in detail below, since it presents the main challenge of this problem by being non-monotone.

In a nutshell, the main result of this article is that a solution to (1) exists for suitable choices of parameters, for nonlinearities which are suitable mollified versions of the sign function,  $\alpha \psi'(u) \approx \alpha \operatorname{sgn}(u)$ .

Mathematically, this equation combines a number of difficulties. It combines a differential operator (the second derivative) with a difference operator ( $\Delta_D$ ). See, e.g., [8] for the subject of such functional equations. Here the equation is looking ‘forward’,  $u(x+1)$ , and ‘backward’,  $u(x-1)$ . The theory of such

advance-delay equations is still not very well developed, though there are very remarkable results, employing tools ranging from variational techniques to centre manifold/normal form analysis, for example [7, 10, 4]. The non-monotonicity of  $g'$  finally is the core difficulty of the problem.

Physically, (1) is the travelling wave equation for the so-called Frenkel-Kontorova model of dislocation dynamics [6]. There, the model proposed is

$$mu_k'' = \beta(u_{k+1} - 2u_k + u_{k-1}) - 2\pi \frac{\alpha}{\gamma} \sin\left(\frac{2\pi}{\gamma} u_k\right) \quad (2)$$

with some constants  $\alpha, \beta, \gamma$ , describing the displacement  $u_k$  of atom  $k \in \mathbb{Z}$  in a one-dimensional chain; the nonlinearity is the derivative of an on-site potential describing the interaction with atoms above and below the chain of atoms considered. The periodicity of the nonlinearity thus reflects the periodic nature of a crystalline lattice. The Frenkel-Kontorova chain is a fundamental model of dislocation dynamics, describing how an imperfection (dislocation) travels through a crystalline lattice; see in particular the survey [3]. The simplest motion that may exist is that of a travelling wave,  $u_j(t) = u(j - ct)$  with wave speed  $c$ . This *ansatz* transforms (2), after rescaling, into (1), with sinusoidal on-site potential  $g$ .

We study the situation where this potential is piecewise quadratic, with small concave parts smoothing out the cusp at the meeting point of two parabola. For piecewise quadratic on-site potentials, there is a long history of formal solutions, going back at least to Atkinson and Cabrera [2]. It has been pointed out that formal calculations often depend on the validity of a sign condition (which will be encountered here as well) [5, 12].

There are few rigorous results for nonconvex interaction potentials available, in particular for heteroclinic solutions as we will study. A very remarkable existence result for such solutions is that of Iooss and Kirchgässner [10]; there a general theory for small solutions is developed. Here we are interested in (large) heteroclinic solutions that stay asymptotically for  $x \rightarrow -\infty$  in one well of a nonconvex on-site potential  $g$  and for  $x \rightarrow \infty$  in another well. For the particular choice  $\alpha\psi'(u) = \alpha \operatorname{sgn}(u)$ , the existence of such travelling waves has been established for suitable parameters with an argument based on Fourier estimates [11]. Here we show that this result holds true in greater generality, in particular for on-site potentials where the concave part is not degenerate as it is assumed in [11]. We work in a nonlinear setting where the Fourier methods of [11] are not applicable.

The existence of heteroclinic travelling waves for the Frenkel-Kontorova problem (2) has been open since 1939 (for coherent spatially localised temporally periodic solutions, existence was established in the seminal paper by MacKay and Aubry [13]; see also [14]). We are presently unable to answer this question for the sinusoidal on-site potential, since we use the explicit knowledge of wave trains in harmonic chains. One interpretation of our result is that it shows that wave trains in one well of  $g$  can be joined to another train in another well, and this transition signifies a moving dislocation. We can establish this

result for a class of smooth potentials which have harmonic wells and small spinodal (concave) regions. Since the potentials we consider are structurally very similar to the original sinusoidal on-site potential, one would expect that existence holds for that potential as well, under similar choices of the parameters made. Yet a proof of this conjecture seems far from straightforward.

We remark that for the Fermi-Pasta Ulam chain with smooth nonconvex interaction potential, a different approach has been employed to prove the existence of heteroclinic waves for cases where the potential has a small spinodal (concave) region [9]. As the method used here, the approach relies on a perturbation argument, but then proceeds differently by relying on the Banach fixed point theorem, following a careful analysis of an integral equation describing the travelling wave equation.

The framework developed in the present article is relatively flexible and allows potentially the analysis of a range of problems in the setting of (at least) the Frenkel-Kontorova chain. To give an example, we study in Section 4 the problem with a piecewise quadratic on-site potential,  $\psi'(u) = \text{sgn}(u)$ , and establish what is to our knowledge the first proof of solutions exhibiting two transitions between the wells of the on-site potential. It can be regarded as a simplified version of the shadowing lemma [1].

## 2 Setup and main result

The central argument we are going to employ is a Schauder fixed point theorem. This is possibly surprising, as equation (1) is defined on the whole real line and therefore there is *a priori* no reason to expect compactness properties for (1). We now sketch the setting in which the Schauder theorem applies.

We start by considering the linear part of (1). The linear operator

$$u \rightarrow Lu = c^2 u'' - \Delta_D u + \alpha u \quad (3)$$

has in Fourier space the representation

$$-c^2 \zeta^2 + 2(1 - \cos \zeta) + \alpha = -c^2 \zeta^2 + 4 \sin^2(\zeta/2) + \alpha =: D(\zeta), \quad (4)$$

where  $D$  is the *dispersion function*. Obviously, for the sound speed,  $c = 1$ , the dispersion relation  $D$  has exactly two nonzero roots  $\pm k_0$ , where

$$k_0 := \frac{\pi}{2} \quad (5)$$

if

$$\alpha = c^2 \left( \frac{\pi}{2} \right)^2 - 2, \quad (6)$$

and furthermore  $D'(\zeta) = -2c^2 \zeta + 2 \sin \zeta$  vanishes only at  $\zeta = 0$ . We will work in a parameter regime where  $c$  is marginally subsonic; we keep  $k_0$  fixed by (5) and  $\alpha$  given by (6). Then  $c$  is the only free parameter in the dispersion relation. Since we seek to find heteroclinic solutions, we will focus on subsonic waves, that is,  $c \leq 1$ .

By continuity, the dispersion function will have exactly two roots near  $\pm k_0$  for ‘near sonic’ subsonic  $c$ .

Our main theorem can be considered as perturbation result of [11], where the special case  $\psi'(u) = \text{sgn}(u)$  is considered. We sketch the situation for this degenerate potential briefly. For  $|\lambda| < 1$  and  $\theta \in [0, 2\pi)$ , trivially  $1 + \lambda \sin(k_0 \cdot + \theta)$  is a solution to (1) on  $[1, \infty)$  and  $-1 + \lambda \sin(k_0 \cdot - \theta)$  is a solution on  $(-\infty, -1]$ . The question is whether these two solution segments can be glued together to form a heteroclinic solution, traversing from one well of the on-site potential  $g$  to another.

The answer is affirmative for the degenerate potential discussed in this paragraph, as shown in [11] (recalled in Theorem 2.1 below). This solution  $u \in H_{\text{loc}}^2(\mathbb{R})$  is odd,  $u(x) = -u(-x)$ , and heteroclinic in the sense that

$$\lim_{x \rightarrow \pm\infty} [u(x) \mp 1 - \lambda \sin(k_0 x \pm \theta)] = 0$$

for some  $\lambda$  and  $\theta$ , and  $\alpha$  given by (6). This solution is well approximated by the explicit function

$$u_{\text{pa}}(x) := \text{sgn}(x) \left[ A \left( 1 - e^{-\beta|z|} \right) + B (1 - \cos(k_0 z)) \right], \quad (7)$$

with

$$A = \frac{c^2 k_0^2 - \alpha}{c^2 (\beta^2 + k_0^2)} \quad \text{and} \quad B = \frac{\alpha + \beta^2 c^2}{c^2 (\beta^2 + k_0^2)} \quad (8)$$

and

$$\beta^2 = \frac{\alpha}{c^2} \cdot \frac{k_0 \sin(k_0)}{2 - 2 \cos(k_0) - k_0 \sin(k_0)} = \frac{\alpha}{c^2} \cdot \frac{k_0}{2 - k_0}.$$

The argument in [11] and this paper uses an idea developed by Schwetlick and Zimmer for a Fermi-Pasta-Ulam chain with nonconvex interaction potential, and no on-site potential [15]. This idea is to represent the solution  $u$  as  $u = u_{\text{p}} - r$  with explicitly given  $u_{\text{p}}$ ; then the analysis is reduced to a careful investigation of the Fourier representation of  $r$ . Here, we will argue similarly and consider a “profile” function  $u_{\text{p}} \in H_{\text{loc}}^2(\mathbb{R})$ . By profile function we mean that the function  $c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}})$  satisfies

$$(1 + x^2)(c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}})) \in L^2(\mathbb{R}) \quad (9)$$

$$\int_{\mathbb{R}} [c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}})] \sin(k_0 \cdot) dx = 0. \quad (10)$$

The former condition implies  $c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha \text{sgn}(u_{\text{p}}) \in L^1(\mathbb{R})$ , so the latter condition is well posed. In addition, the function should be odd,  $\text{sgn}(u_{\text{p}}(x)) = \text{sgn}(x)$  on  $\mathbb{R}$ , vanishes at  $x = 0$ , satisfy  $u_{\text{p}}'(0) > 0$  and  $\liminf_{|x| \rightarrow \infty} |u_{\text{p}}(x)| > 0$ , so that equation (14) below holds.

It is somewhat tedious but not difficult to find such a  $u_{\text{p}}$ . Specifically, we could use the profile function  $u_{\text{pa}}$  given above. However, we will use the solution to (1) with the special force  $\psi'(x) = \text{sgn}(x)$  as profile. We therefore recall the existence result for this function.

**Theorem 2.1** ([11, Theorem 4.1]). *Let  $\psi'(x) = \text{sgn}(x)$ . Let  $c$  be such that  $c^2 \in [0.83, 1]$ . Let  $k_0$  be given by (5) and  $\alpha$  be given by (6). Then (1) has a solution  $u = u_{\text{pa}} - r$  with  $u_{\text{pa}}$  given by (7) with*

$$\sqrt{\frac{\pi}{2}} |r(z)| \leq \begin{cases} 0.257 & \text{for } c^2 \in [0.9, 1], \\ 0.339 & \text{for } c^2 \in [0.83, 0.9], \end{cases}$$

and

$$\sqrt{\frac{\pi}{2}} |r'(z)| \leq \begin{cases} 0.43 & \text{for } c^2 \in [0.9, 1], \\ 0.34 & \text{for } c^2 \in [0.83, 0.9]. \end{cases}$$

So below  $u_{\text{p}}$  will be the function  $u$  of Theorem 2.1. We are left with having to find  $r \in H_{\text{odd,loc}}^2(\mathbb{R})$  (that is,  $r \in H_{\text{loc}}^2(\mathbb{R})$  and  $r(-x) = -r(x)$ ) such that  $u_{\text{p}} - r$  is a solution:

$$c^2(u_{\text{p}} - r)'' - \Delta_D(u_{\text{p}} - r) + \alpha(u_{\text{p}} - r) - \alpha\psi'(u_{\text{p}} - r) = 0,$$

and hence for  $r$

$$c^2 r'' - \Delta_D r + \alpha r = c^2 u_{\text{p}}'' - \Delta_D u_{\text{p}} + \alpha u_{\text{p}} - \alpha\psi'(u_{\text{p}} - r),$$

which is an equation of the form

$$c^2 r'' - \Delta_D r + \alpha r = Q,$$

or  $Lr = Q$  with nonlinear  $Q$ . We will employ Schauder's fixed point theorem to establish a solution to this equation. The main result can be stated as follows.

**Theorem 2.2.** *For  $\epsilon > 0$ , let the even function  $\psi = \psi_{\epsilon} \in C^2(\mathbb{R})$  be such that  $\psi'_{\epsilon}(x) = \text{sgn}(x)$  for  $|x| \geq \epsilon$  and  $|\psi''_{\epsilon}(x)| \leq 2\epsilon^{-1}$  for  $|x| < \epsilon$ . Let  $k_0$  be given by (5),  $\alpha$  be given by (6). Then there exists a range of subsonic velocities  $c$  close to 1 such that for these velocities, there exists a heteroclinic solution to (1).*

We remark that one of the conditions imposed on closeness of  $c$  to 1 is  $c^2 \in [0.83, 1]$  as only in this case we can build on the existence result Theorem 2.1.

Theorem 2.2 is proved in the next section. We state one auxiliary statement for the equation  $Lr = Q$ .

**Proposition 2.3.** *If  $Q \in L_{\text{odd}}^2(\mathbb{R})$  satisfies*

$$(1 + x^2)Q \in L^2(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0,$$

*then, for all  $c$  near enough to 1, there exists a unique function  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $Lr = c^2 r'' - \Delta_D r + \alpha r = Q$ . Moreover*

$$\|r\|_{H^2(\mathbb{R})} := \|(1 + k^2)\hat{r}\|_{L^2(\mathbb{R})} \leq \{C_1 + ((4 + \alpha)C_1 + 1)/c^2\} \|(1 + x^2)Q\|_{L^2(\mathbb{R})}$$

*for some constant  $C_1 > 0$  (independent of  $c$  near 1).*

An extension of this result to functions  $Q$  which are not necessarily odd can be found Proposition A.1 in the Appendix.

*Proof.* The assumptions imply that  $\widehat{Q} \in H^2(\mathbb{R}, \mathbb{C})$ ,  $\widehat{Q}(\pm k_0) = 0$  and that there exists a unique  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $c^2 r'' - \Delta_D r + \alpha r = Q$ , namely

$$\widehat{r}(k) = \frac{\widehat{Q}(k)}{D(k)}, \text{ for } k \in \mathbb{R}.$$

As  $Q$  is odd and real-valued,  $i\widehat{Q}$  is odd and real-valued. Therefore so are  $i\widehat{r}$  and  $r$ . Moreover,

$$\begin{aligned} \|\widehat{Q}'\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{|x|}{1+x^2} (1+x^2) |Q(x)| dx \\ &\leq \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} \frac{x^2}{(1+x^2)^2} dx \right)^{1/2} \|(1+x^2)Q\|_{L^2(\mathbb{R})} = \frac{1}{2} \|(1+x^2)Q\|_{L^2(\mathbb{R})}, \end{aligned}$$

(note that  $(1/2) \arctan x - (1/2)x/(1+x^2)$  is a primitive of  $x^2(1+x^2)^{-2}$ ).

Consider for a while  $c = 1$ . For  $|k| \in [k_0/2, 3k_0/2] \setminus \{k_0\}$ , one gets by Cauchy's mean value theorem applied to the real-valued functions  $i\widehat{Q}$  and  $D$

$$\left| \frac{\widehat{Q}(k)}{D(k)} \right| \leq \sup_{|s| \in [k_0/2, 3k_0/2] \setminus \{k_0\}} \left| \frac{\widehat{Q}'(s)}{D'(s)} \right| \leq |D'(k_0/2)|^{-1} \frac{1}{2} \|(1+x^2)Q\|_{L^2(\mathbb{R})}.$$

For  $|k| \notin [k_0/2, 3k_0/2]$ , one gets  $|D(k)| \geq \min\{|D(k_0/2)|, |D(3k_0/2)|\}$ . Hence

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\widehat{Q}(k)}{D(k)} \right|^2 dk &\leq \max\{|D(k_0/2)|^{-2}, |D(3k_0/2)|^{-2}\} \int_{|k| \notin [k_0/2, 3k_0/2]} |\widehat{Q}(k)|^2 dk \\ &\quad + 2k_0 |D'(k_0/2)|^{-2} \frac{1}{4} \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2 \\ &\leq \left( \max\{|D(k_0/2)|^{-2}, |D(3k_0/2)|^{-2}\} + \frac{1}{2} k_0 |D'(k_0/2)|^{-2} \right) \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2 \\ &= C_1^2 \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

This estimate remains valid for all  $c$  close to 1 if we first increase slightly  $C_1$ . As a consequence

$$c^2 \|r''\|_{L^2(\mathbb{R})} \leq (4+\alpha) \|r\|_{L^2(\mathbb{R})} + \|Q\|_{L^2(\mathbb{R})} \leq ((4+\alpha)C_1+1) \|(1+x^2)Q\|_{L^2(\mathbb{R})}$$

and

$$\begin{aligned} \|r\|_{H^2(\mathbb{R})} &= \|(1+k^2)\widehat{r}\|_{L^2(\mathbb{R})} \leq \|r\|_{L^2(\mathbb{R})} + \|r''\|_{L^2(\mathbb{R})} \\ &\leq \{C_1 + ((4+\alpha)C_1+1)/c^2\} \|(1+x^2)Q\|_{L^2(\mathbb{R})}. \end{aligned}$$

□

### 3 Proof of Theorem 2.2

#### 3.1 Preliminaries

We now turn to the proof of Theorem 2.2. We seek a solution to (1),

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \psi'(u) = 0.$$

By assumption,  $\psi \in C^2(\mathbb{R})$  is even and for its derivative it holds that  $\psi' = \text{sgn}$  outside a bounded set. We split the solution  $u$  sought to (1) as

$$u = u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r, \quad (11)$$

where the profile function  $u_p \in H_{\text{loc}}^2(\mathbb{R})$  is odd and satisfies properties (9) and (10). Further,  $\gamma \in \mathbb{R}$  is assumed to be sufficiently close to 0, and  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  is an odd function such that for each  $l = 0, 1, 2$ ,

$$(1 + x^2) \frac{d^l}{dx^l} (u_o(x) - \text{sgn}(x) \cos(k_0 x)) \in L^2(\mathbb{R} \setminus [-1, 1]). \quad (12)$$

For example, one can choose  $u_o$  to agree with  $\text{sgn}(x) \cos(k_0 x)$  outside a bounded interval. It is not hard to give an explicit representation for  $u_o$ , whereas  $u_p$  is the solution given by Theorem 2.1; the task is then to find the corrector  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $u$  as in (11) solves (1). The periodic term  $\gamma \sin(k_0 \cdot)$  is separated from  $u_p$  for mere convenience; obviously this term could be added to  $u_p$  and then  $\tilde{u}_p := u_p + \gamma \sin(k_0 \cdot)$  satisfies (9) and (10) and could replace  $u_p$ .

With this notation, we can now restate Theorem 2.2 in a more detailed form we are going to establish.

**Theorem 3.1.** *For  $\epsilon > 0$ , let the even function  $\psi = \psi_\epsilon \in C^2(\mathbb{R})$  be such that  $\psi'_\epsilon(x) = \text{sgn}(x)$  for  $|x| \geq \epsilon$  and  $|\psi''_\epsilon(x)| \leq 2\epsilon^{-1}$  for  $|x| < \epsilon$ . Let  $k_0$  be given by (5),  $\alpha$  be given by (6). Then there exists a range of subsonic velocities  $c$  with  $c^2 \geq 0.83$  such that a heteroclinic solution to (1) exists, in the following sense. Let the odd function  $u_p \in H_{\text{loc}}^2(\mathbb{R})$  be the solution to the equation  $c^2 u'' - \Delta_D u + \alpha u - \alpha \text{sgn}(u) = 0$  of Theorem 2.1, and let the odd function  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  satisfy (12).*

*Then for all  $|\gamma|$  and  $\rho > 0$  small enough, there exists  $\epsilon_0 > 0$  satisfying the following property. For every  $\epsilon \in (0, \epsilon_0)$ , there exists  $r \in H_{\text{odd}}^2(\mathbb{R})$  and  $\beta \in \mathbb{R}$  such that  $\|r\|_{H^2(\mathbb{R})} < \rho$  and  $u := u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r$  is a solution to (1),*

$$\begin{aligned} & c^2 (u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r)'' - \Delta_D (u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) \\ & + \alpha (u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) - \alpha \psi'_\epsilon(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) = 0. \end{aligned} \quad (13)$$

Theorem 2.2 follows immediately once Theorem 3.1 is established, and the rest of the article is devoted to the proof of Theorem 3.1.

We start the proof by considering the linear operator  $L$  of (3) with  $\alpha$  as in (6) and  $c$  being slightly subsonic. Specifically, we first study the equation  $Lr = Q$  under the hypothesis  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0$ , with  $k_0 = \pi/2$ . Roughly speaking, in the equation  $Lr = Q$ , the right-hand side is replaced by a new expression  $Q$  depending on  $u_o$  and a real parameter  $\beta$  chosen so that  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0$ .

**Lemma 3.2.** *Let  $u_p$  be the solution to the special case  $\psi'(x) = \text{sgn}(x)$  recalled in Theorem 2.1. There exists  $\rho > 0$  such that, for all  $r$  in the ball  $\overline{B(0, \rho)} \subset H_{\text{odd}}^2(\mathbb{R})$ ,  $\text{sgn}(u_p(x) - r(x)) = \text{sgn}(x)$  on  $\mathbb{R}$ .*

*Proof.* Recall the Sobolev estimates

$$\begin{aligned} \|r\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1+k^2} (1+k^2) |\widehat{r}| dk \\ &\leq \frac{1}{\sqrt{2\pi}} \sqrt{\int_{\mathbb{R}} \frac{1}{(1+k^2)^2} dk} \|r\|_{H^2(\mathbb{R})} = \frac{1}{2} \|r\|_{H^2(\mathbb{R})} \end{aligned}$$

and

$$\begin{aligned} \|r'\|_{L^\infty(\mathbb{R})} &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{|k|}{1+k^2} (1+k^2) |\widehat{r}| dk \\ &\leq \frac{1}{\sqrt{2\pi}} \sqrt{\int_{\mathbb{R}} \frac{k^2}{(1+k^2)^2} dk} \|r\|_{H^2(\mathbb{R})} = \frac{1}{2} \|r\|_{H^2(\mathbb{R})} . \end{aligned}$$

By symmetry, it suffices to consider positive  $x$ . Hence it suffices to choose  $\rho_0 > 0$  such that there is a point  $x_0 \in (0, 1]$  such that

$$u_p(x) > \rho_0/2 \text{ for } x > x_0 \text{ and } u_p'(x) > \rho_0/2 \text{ for every } x \in [0, x_0]. \quad (14)$$

Since  $u_p$  satisfies this property for some  $\rho_0$ , so the claim follows for any  $\rho \in (0, \rho_0)$ .  $\square$

Throughout this article, we will assume  $\rho \in (0, \rho_0)$ . We also assume that  $\epsilon < \rho_0/6$ , so that  $\psi'(s) = \text{sgn}(s)$  for all  $|s| \geq \rho_0/6$ .

If we add the requirement on  $\beta$ ,  $\gamma$  and  $r$  that the condition

$$|\beta u_o(x) + \gamma \sin(k_0 x) - r(x)| \leq \frac{2}{3} |u_p(x)|$$

is fulfilled for all  $x \in \mathbb{R}$ , the solving (1) with the *ansatz* (11) is equivalent to solving

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \partial_1 \Psi(u, x) = 0 \text{ with } u = u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r, \quad (15)$$

where  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies

$$\begin{cases} \Psi(u, x) = \psi(u) & \text{for } |x| \leq 1, \\ \Psi(u, x) = \text{sgn}(x)u & \text{for } |x| \geq 1. \end{cases}$$

We prove the existence of a solution using Schauder's fixed point theorem.

### 3.2 Application of Schauder's fixed point theorem

In this section, we prove the existence of a solution of a slightly relaxed problem, Equation (20) below, under fairly abstract assumptions, notably (C1), (C2) in



Theorem 3.5 below. The following sections then establish that the original problem can be cast in the setting studied here.

Specifically, consider a modification (13) for  $r \in H_{\text{odd}}^2(\mathbb{R})$  and  $\beta \in \mathbb{R}$ , and recall  $\psi'(u(x)) = \partial_1 \Psi(u(x), x)$  for the function  $u$  we have in mind,

$$\begin{aligned} & c^2(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r)'' - \Delta_D(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) \\ & + \alpha(u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r) - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, x) = 0; \end{aligned} \quad (16)$$

here the new ingredient is a function  $\xi \in C^1(\mathbb{R})$  with  $\|\xi'\|_{L^\infty(\mathbb{R})} < \infty$ . Thus, in a first step, we replaced  $\beta$  by  $\xi(\beta)$  in the nonlinear term. As  $\xi'$  is assumed to be bounded, the function  $\xi$  allows us to control the nonlinear term without restrictions on the size of  $\beta$ . In a second step, we shall assume that  $\xi$  is the identity near 0 and show that the relevant values of  $\beta$  are sufficiently close to 0, so that  $\xi(\beta) = \beta$  for these values of  $\beta$ .

The assumptions in this Subsection are as follows. We recall  $k_0$  is given by (5),  $\alpha$  is given by (6), and  $c$  is close to 1. We have seen that then the dispersion function in (4) has exactly two simple roots  $\pm k_0$ . Furthermore, for the linear operator given in (3),  $L \sin(k_0 \cdot) = 0$ . Let  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  be of class  $C^2$  with respect to the first variable,  $\Psi$ ,  $\partial_1 \Psi$  and  $\partial_{11}^2 \Psi$  be measurable with respect to the second variable,  $\partial_1 \Psi$  be odd and

$$|\partial_{11}^2 \Psi(s, x)| \leq \frac{\mu}{(1+x^2)^{3/2}} \quad (17)$$

for some constant  $\mu > 0$ . Note that

$$(1+x^2) \frac{1}{(1+x^2)^{3/2}} \in L^2(\mathbb{R}).$$

The size of  $\mu$  does not matter in what follows (in particular, it is not assumed to be small).

We recall that the parameter  $\gamma$  is real-valued, and that  $u_p$  is a given odd function in  $H_{\text{loc}}^2(\mathbb{R})$  satisfying

$$\begin{aligned} & \sup_{\beta \in \mathbb{R}} \left\| (1+x^2)^{3/2} \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p \right. \right. \\ & \quad \left. \left. - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x), x) \right) \right\|_{L^\infty(\mathbb{R})} < \infty. \end{aligned} \quad (18)$$

The odd function  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  satisfies (12). Thus, since  $L \cos(k_0 \cdot) = 0$ ,

$$(1+x^2) L u_o = (1+x^2) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \in L_{\text{odd}}^2(\mathbb{R}).$$

It follows that the map

$$\begin{aligned} (r, \beta) \rightarrow \Gamma(r, \beta) = & (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p \right. \\ & \left. - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x) - r, x) \right) \in L_{\text{odd}}^2(\mathbb{R}) \end{aligned}$$

is well-defined on  $H_{\text{odd}}^2(\mathbb{R}) \times \mathbb{R}$  and of class  $C^1$ .

**Lemma 3.3.** *The map  $\Gamma: H_{\text{odd}}^2(\mathbb{R}) \times \mathbb{R} \rightarrow L_{\text{odd}}^2(\mathbb{R})$  is compact.*

*Proof.* The map can be written as

$$\begin{aligned} \Gamma(r, \beta) = & (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x), x) \right) \\ & + \alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 x) - sr, x) r \, ds, \end{aligned}$$

which is the sum of two terms in  $L^2(\mathbb{R})$  (see (17) and (18)). Let  $\{(r_n, \beta_n)\} \subset H_{\text{odd}}^2(\mathbb{R}) \times \mathbb{R}$  be a bounded sequence. We verify that  $\{\Gamma(r_n, \beta_n)\}$  has a Cauchy subsequence in  $L_{\text{odd}}^2(\mathbb{R})$ . Let  $\varepsilon > 0$ .

Since  $\xi$  is continuous on  $\mathbb{R}$ , the sequence  $\{\xi_n\} := \{\xi(\beta_n)\}$  is bounded. Taking a convergent subsequence  $\{\xi_{n_k}\}$ , equation (18) and the dominated convergence theorem ensure that the first term of  $\Gamma(r_{n_k}, \beta_{n_k})$  converges as  $k \rightarrow \infty$ . Hence, for  $k, l$  large enough,

$$\begin{aligned} & \left\| (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta_{n_k}) u_o + \gamma \sin(k_0 x), x) \right) \right. \\ & \left. - (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta_{n_l}) u_o + \gamma \sin(k_0 x), x) \right) \right\|_{L^2(\mathbb{R})} < \frac{\varepsilon}{2}. \end{aligned}$$

To deal with the second term, we split  $\mathbb{R}$  in two parts, namely  $I_\varepsilon := [-x_\varepsilon, x_\varepsilon]$  and its complement in  $\mathbb{R}$ , where  $x_\varepsilon > 0$  is large. The motivation for this split is that many Sobolev embeddings are compact on an bounded interval, whereas the second term can be assumed as small as needed when restricted to the complement of  $I_\varepsilon$ . More precisely, given  $\varepsilon > 0$ , choose  $x_\varepsilon$  large enough so that for all  $k$

$$\alpha \left\| (1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_k} u_o + \gamma \sin(k_0 x) - sr_{n_k}, x) r_{n_k} \, ds \right\|_{L^2(\mathbb{R} \setminus I_\varepsilon)} < \frac{\varepsilon}{8}$$

(see (17)). Using the compact embedding  $H^2(-x_\varepsilon, x_\varepsilon) \subset C[-x_\varepsilon, x_\varepsilon]$ , by taking a further subsequence if necessary, we can assume that  $\{r_{n_k}\}$  converges in  $C[-x_\varepsilon, x_\varepsilon]$ . It follows, again from the dominated convergence theorem, that

$$\alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_k} u_o + \gamma \sin(k_0 x) - sr_{n_k}, x) r_{n_k} \, ds$$

converges in  $L^2(-x_\varepsilon, x_\varepsilon)$ . Hence, for  $k, l$  large enough,

$$\begin{aligned} & \left\| \alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_k} u_o + \gamma \sin(k_0 x) - sr_{n_k}, x) r_{n_k} \, ds \right. \\ & \left. - \alpha(1+x^2) \int_0^1 \partial_{11}^2 \Psi(u_p + \xi_{n_l} u_o + \gamma \sin(k_0 x) - sr_{n_l}, x) r_{n_l} \, ds \right\|_{L^2(\mathbb{R})} < \varepsilon/2. \end{aligned}$$

Thus  $\{\Gamma(r_{n_k}, \beta_{n_k})\}$  is a Cauchy subsequence.  $\square$

By (26) of Proposition A.2 in the Appendix,

$$\int_{\mathbb{R}} (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \sin(k_0 \cdot) dx = -2c^2 k_0 + 2 < 0$$

if  $c > k_0^{-1/2}$ . Assume that, for all  $r$  in some subset of  $H_{\text{odd}}^2(\mathbb{R})$  and all  $\beta \in \mathbb{R}$ ,

$$\begin{aligned} & \left| \int_{\mathbb{R}} \alpha \partial_{11}^2 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \xi'(\beta)u_o \sin(k_0 \cdot) dx \right| \\ & \leq C \left| \int_{\mathbb{R}} (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \sin(k_0 \cdot) dx \right| = C 2(c^2 k_0 - 1) \end{aligned}$$

for some constant  $C \in [0, 1)$ . Then for fixed  $r$  in the given subset, the equation

$$\begin{aligned} & \int_{\mathbb{R}} \left( c^2 (u_p + \beta u_o + \gamma \sin(k_0 \cdot))'' - \Delta_D (u_p + \beta u_o + \gamma \sin(k_0 \cdot)) + \alpha (u_p + \beta u_o + \gamma \sin(k_0 \cdot)) \right. \\ & \quad \left. - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, x) \right) \sin(k_0 \cdot) dx = 0 \end{aligned}$$

can uniquely be solved for  $\beta$  as a  $C^1$ -function of  $r$ ,  $\beta = \beta(r)$ , thanks to Banach's fixed point theorem and the implicit function theorem.

**Lemma 3.4.** *The map  $r \rightarrow \beta(r)$  is bounded on bounded sets.*

*Proof.* The proof of Lemma 3.3 shows an additional property, namely that the map  $(r, \beta) \rightarrow \Gamma(r, \beta)$  is bounded on every set on which the  $r$ -component is bounded. As a consequence, by definition of  $\beta = \beta(r)$ ,

$$\begin{aligned} 2(c^2 k_0 - 1)\beta &= -\beta \int_{\mathbb{R}} (c^2 u_o - \Delta_D u_o + \alpha u_o) \sin(k_0 x) dx = \\ &= \int_{\mathbb{R}} [c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot)] \sin(k_0 x) dx \end{aligned}$$

and

$$\beta = \frac{\int_{\mathbb{R}} [c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot)] \sin(k_0 x) dx}{2(c^2 k_0 - 1)}. \quad (19)$$

The map  $r \rightarrow \beta(r) = \frac{1}{2}(c^2 k_0 - 1)^{-1} \int_{\mathbb{R}} \Gamma(r, \beta(r))(1 + x^2)^{-1} \sin(k_0 x) dx$  is thus bounded on bounded sets.  $\square$

Hence the problem can be written as  $c^2 r'' - \Delta_D r + \alpha r = Q$ , with

$$\begin{aligned} Q &= c^2 (u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot))'' - \Delta_D (u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot)) \\ &+ \alpha (u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot)) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \\ &= \beta(r) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) + (1 + x^2)^{-1} \Gamma(r, \beta(r)) \in L_{\text{odd}}^2(\mathbb{R}) \end{aligned}$$

and  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0$  by definition of  $\beta(r)$ . On the other hand, if  $Q \in L^2(\mathbb{R})$  is odd with

$$(1+x^2)Q \in L^2(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = 0,$$

we saw in Proposition 2.3 that there exists a unique odd  $r = L^{-1}Q \in H^2(\mathbb{R})$  such that  $Lr = c^2 r'' - \Delta_D r + \alpha r = Q$ . Moreover

$$\|L^{-1}Q\|_{H^2(\mathbb{R})} = \|r\|_{H^2(\mathbb{R})} \leq \{C_1 + ((4+\alpha)C_1 + 1)/c^2\} \|(1+x^2)Q\|_{L^2(\mathbb{R})}$$

for some constant  $C_1 > 0$ .

The problem (16) studied in this Subsection can be written as

$$\begin{aligned} r = L^{-1}Q = L^{-1} \Big( & c^2(u_p + \beta u_o + \gamma \sin(k_0 \cdot))'' - \Delta_D(u_p + \beta u_o + \gamma \sin(k_0 \cdot)) \\ & + \alpha(u_p + \beta u_o + \gamma \sin(k_0 \cdot)) - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, x) \Big) \end{aligned} \quad (20)$$

with  $\beta = \beta(r)$ .

**Theorem 3.5.** *Let  $\xi$  be in  $C^1(\mathbb{R})$  with  $\|\xi'\|_{L^\infty(\mathbb{R})} < \infty$ . Let  $k_0$  be as in (5) and  $\alpha$  given by (6), Let  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  be of class  $C^2$  with respect to the first variable, let  $\Psi$ ,  $\partial_1 \Psi$  and  $\partial_{11}^2 \Psi$  be measurable with respect to the second variable, and  $\partial_1 \Psi$  be odd. Assume that the hypotheses (12), (17) and (18) hold. Suppose that there exists an open ball  $B(0, \rho) \subset H_{\text{odd}}^2(\mathbb{R})$  such that*

$$\sup_{r \in \overline{B(0, \rho)}, \beta \in \mathbb{R}} \left| \int_{\mathbb{R}} \alpha \partial_{11}^2 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \xi'(\beta)u_o \sin(k_0 \cdot) dx \right| < 2(c^2 k_0 - 1) \quad (C1)$$

and

$$\sup_{r \in \overline{B(0, \rho)}} \|(1+x^2)Q(r)\|_{L^2(\mathbb{R})} < \{C_1 + ((4+\alpha)C_1 + 1)/c^2\}^{-1} \rho. \quad (C2)$$

Then there exists a solution  $r \in B(0, \rho)$  to (20).

*Proof.* For all  $r \in \overline{B(0, \rho)}$ ,  $Q = Q(r)$  is well defined with values in

$$Z = \left\{ f \in L_{\text{odd}}^2(\mathbb{R}) : (1+x^2)f \in L^2(\mathbb{R}), \int_{\mathbb{R}} f(x) \sin(k_0 x) dx = 0 \right\}$$

and completely continuous in  $r$  (that is, continuous and compact). The map  $r \rightarrow L^{-1}Q(r)$  sends  $\overline{B(0, \rho)}$  into  $B(0, \rho)$  and is completely continuous. The Schauder fixed point theorem gives a solution  $r \in \overline{B(0, \rho)}$  to the equation  $r = L^{-1}Q(r)$ , and in fact  $r \in B(0, \rho)$ .  $\square$

### 3.3 On the verification of condition (C2)

In this section, we establish one condition, (C2') below, for the verification of condition (C2) in Theorem 3.5. This simpler condition will then be shown in Subsection 3.4 to hold under the assumptions of Theorem 3.1.

By the formula (19) for  $\beta = \beta(r)$  and

$$Q = (c^2 u_o'' - \Delta_D u_o + \alpha u_o) \beta + c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot),$$

one has

$$\begin{aligned} & \|(1+x^2)Q\|_{L^2(\mathbb{R})} \leq \|(1+x^2)(c^2 u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \\ & \times \left| \frac{\int_{\mathbb{R}} \left\{ c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right\} \sin(k_0 x) dx}{2(c^2 k_0 - 1)} \right| \\ & + \|(1+x^2)(c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot))\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence condition (C2) is ensured by the following condition

$$\begin{aligned} & \sup_{r \in \overline{B(0, \rho)}} \left\{ \|(1+x^2)(c^2 u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \frac{\|(1+x^2)^{-1} \sin(k_0 \cdot)\|_{L^2(\mathbb{R})}}{2(c^2 k_0 - 1)} + 1 \right\} \\ & \times \|(1+x^2)(c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot))\|_{L^2(\mathbb{R})} \\ & < \frac{1}{C_1 + ((4 + \alpha)C_1 + 1)/c^2} \rho, \end{aligned}$$

which in turn is ensured by the condition

$$\begin{aligned} & \sup_{r \in \overline{B(0, \rho)}} \left\| (1+x^2) \left( c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \partial_1 \Psi(u_p + \xi(\beta) u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right) \right\|_{L^2(\mathbb{R})} \\ & < \frac{\{C_1 + ((4 + \alpha)C_1 + 1)/c^2\}^{-1} \rho}{\|(1+x^2)(c^2 u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \sqrt{\pi/8}(c^2 k_0 - 1)^{-1} + 1}. \end{aligned}$$

If  $u_p$  is a particular solution to the “unperturbed” equation  $c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha S(u_p, \cdot) = 0$  for some function  $S$ , if

$$\begin{aligned} & \|(1+x^2)(\alpha S(u_p, \cdot) - \alpha \partial_1 \Psi(u_p, \cdot))\|_{L^2(\mathbb{R})} \\ & + \sup_{r \in \overline{B(0, \rho)}} \left\| (1+x^2) \left( \alpha \partial_1 \Psi(u_p, \cdot) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r)) u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right) \right\|_{L^2(\mathbb{R})} \\ & < \frac{(C_1 + ((4 + \alpha)C_1 + 1)/c^2)^{-1} \rho}{\|(1+x^2)(c^2 u_o'' - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \sqrt{\pi/8}(c^2 k_0 - 1)^{-1} + 1} \quad (\text{C2}') \end{aligned}$$

and if the condition (C1) holds true, then the “perturbed” problem, in which  $S$  is replaced by  $\partial_1 \Psi$  and the parameter  $\gamma$  can be chosen in  $\mathbb{R}$ , has a solution  $r \in B(0, \rho)$ .

### 3.4 Verification of the conditions in Theorem 3.5

In this section, we prove Theorem 3.1. We have to show that the assumptions made there imply those of Theorem 3.5, and show that  $\xi$  can be chosen to be the identity in the region of interest.

We make the same assumptions on  $k_0$ ,  $\alpha$ ,  $u_o$  and  $u_p$  as in Theorem 3.1. In particular, the chosen  $u_p$  is such that  $u_p'(0) > 0$ ,

$$\int_{\mathbb{R}} (c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \operatorname{sgn}(u_p)) \sin(k_0 \cdot) dx = 0,$$

and

$$\left\| (1 + x^2)^{3/2} (c^2 u_p'' - \Delta_D u_p + \alpha u_p - \alpha \operatorname{sgn}(u_p)) \right\|_{L^\infty(\mathbb{R})} < \infty.$$

Let  $\rho_0 > 0$  satisfy (14); then  $|u_p(x)| > \rho_0/2$  for all  $|x| \geq 1$ .

**Lemma 3.6.** *In the setting of this subsection,  $\xi$  can be chosen such that the solution given by Theorem 3.5 solves (13).*

*Proof.* In Equation (16), we choose  $\xi$  such that it is the identity function in a neighbourhood of  $\beta = 0$  and

$$\|\xi\|_{L^\infty(\mathbb{R})} |u_o(x)| \leq \frac{1}{3} |u_p(x)| \text{ for all } x \in \mathbb{R}.$$

If  $|\gamma|$  and  $\|r\|_{H^2(\mathbb{R})}$  are small enough, then for every  $x \in \mathbb{R}$

$$|u_p(x) + \xi(\beta)u_o(x) + \gamma \sin(k_0 x) - r(x)| \geq \frac{1}{3} |u_p(x)| \quad (21)$$

and thus

$$\partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, x) = \psi'(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r).$$

Hence, we will obtain the solution  $u = u_p + \beta u_o + \gamma \sin(k_0 \cdot) - r$  to

$$c^2 u'' - \Delta_D u + \alpha u - \alpha \psi'(u) = 0$$

if, in addition,  $\xi(\beta) = \beta$ . □

**Lemma 3.7.** *Under the assumptions of Theorem 3.1, assumption (17) of Theorem 3.5 holds.*

*Proof.* This is immediate; recall that  $\psi \in C^2(\mathbb{R})$  is even, with  $\psi'(s) = \operatorname{sgn}(s)$  outside a bounded set. By reducing  $\epsilon$  if necessary we can assume that  $\psi'(s) = \operatorname{sgn}(s)$  for all  $|s| \geq \rho_0/6$ . Then  $\Psi: \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies  $\Psi(u, x) = \psi(u)$  for  $|x| \leq 1$  and  $\Psi(u, x) = \operatorname{sgn}(x)u$  for  $|x| \geq 1$ . □

**Lemma 3.8.** *Under the assumptions of Theorem 3.1, the assumptions (18), (C1) and (C2') hold.*

*Proof.* We first establish the claim for (C1). Let us recall that  $\psi$  such that  $|\psi''(s)| \leq 2\epsilon^{-1}$  for  $|s| < \epsilon$  and  $\psi''(s) = 0$  otherwise, where  $\epsilon > 0$ . If  $\epsilon$  is small enough and  $|x| = 6\epsilon/u'_p(0)$ , then

$$|u_p(x)| = u'_p(0) |x| (1 + o(x)) \geq \frac{1}{2} u'_p(0) |x| \geq 3\epsilon$$

and thus  $|u_p(x)| \geq 3\epsilon$  for all  $|x| \geq 6\epsilon/u'_p(0)$  if  $\epsilon$  is small enough. Hence

$$\psi''(u_p(x) + \xi(\beta)u_o(x) + \gamma \sin(k_0 x) - r(x)) = 0$$

for all  $|x| \geq 6\epsilon/u'_p(0)$  if  $|\gamma|, \|r\|_{H^2(\mathbb{R})}$  and  $\epsilon$  are small enough (see (21)). Therefore

$$\begin{aligned} & \left| \int_{\mathbb{R}} \alpha \psi''(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r) \xi'(\beta) u_o \sin(k_0 \cdot) dx \right| \\ & \leq \int_{-6\epsilon/u'_p(0)}^{6\epsilon/u'_p(0)} \alpha 2\epsilon^{-1} |\xi'(\beta) u_o \sin(k_0 \cdot)| dx \\ & \leq \alpha 2\epsilon^{-1} \|\xi'(\beta) u_o\|_{L^\infty(\mathbb{R})} \int_{-6\epsilon/u'_p(0)}^{6\epsilon/u'_p(0)} |k_0 x| dx \\ & \leq \alpha 2\epsilon^{-1} \|\xi'(\beta) u_o\|_{L^\infty(\mathbb{R})} k_0 (6\epsilon/u'_p(0))^2 \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ , uniformly in  $\beta \in \mathbb{R}$  and  $r \in \overline{B(0, \rho)}$  if  $|\gamma|$  and  $\rho > 0$  are small enough. Hence (C1) holds true. Assumption (18) can be verified similarly.

We now show that (C2') is satisfied. We choose for  $u_p$  the solution of the degenerate problem  $c^2 u'' - \Delta_D u + \alpha u - \alpha \text{sgn}(u) = 0$ , see Theorem 2.1, and choose  $\epsilon > 0$  small enough so that

$$\begin{aligned} & \|(1+x^2)(\alpha \text{sgn}(u_p) - \alpha \partial_1 \Psi(u_p, \cdot))\|_{L^2(\mathbb{R})} \\ & < \frac{(\{C_1 + ((4+\alpha)C_1 + 1)/c^2\}^{-1} \rho}{2 \|(1+x^2)(c^2 u''_o - \Delta_D u_o + \alpha u_o)\|_{L^2(\mathbb{R})} \sqrt{\pi/8}(c^2 k_0 - 1)^{-1} + 1}. \end{aligned}$$

Then observe that, for all  $r \in \overline{B(0, \rho)}$ ,

$$\begin{aligned} & \|(1+x^2)(\alpha \partial_1 \Psi(u_p, \cdot) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r, \cdot))\|_{L^2(\mathbb{R})} \\ & \leq \left\| (1+x^2) \alpha \sup_{\lambda \in [0,1]} |\partial_{11}^2 \Psi(u_p + \lambda \xi(\beta(r))u_o + \lambda \gamma \sin(k_0 \cdot) - \lambda r, \cdot)| \right. \\ & \quad \times \left. \|\xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r\| \right\|_{L^2(\mathbb{R})}. \end{aligned}$$

Arguing as above,

$$\begin{aligned} & \left\| (1+x^2)(\alpha \partial_1 \Psi(u_p, \cdot) - \alpha \partial_1 \Psi(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r, \cdot)) \right\|_{L^2(\mathbb{R})} \\ & \leq \alpha 2\epsilon^{-1} \left\| (1+x^2)(\xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r) \right\|_{L^2([-6\epsilon/u'_p(0), 6\epsilon/u'_p(0)])} \rightarrow 0 \end{aligned}$$

as  $\epsilon \rightarrow 0$ , uniformly in  $r \in \overline{B(0, \rho)}$  if  $|\gamma|$  and  $\rho > 0$  are small enough.

By Theorem 3.5, there exists  $r \in H_{\text{odd}}^2(\mathbb{R})$  such that  $\|r\|_{H^2(\mathbb{R})} < \rho$  and

$$\begin{aligned} & c^2(u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot) - r)'' - \Delta_D(u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot) - r) \\ & + \alpha(u_p + \beta(r)u_o + \gamma \sin(k_0 \cdot) - r) - \alpha \psi'(u_p + \xi(\beta(r))u_o + \gamma \sin(k_0 \cdot) - r) = 0. \end{aligned}$$

We also get that  $\beta(r)$  belongs to the neighbourhood of 0 on which  $\xi$  is the identity if  $|\gamma|, \rho, \epsilon$  are small enough. Indeed, by (19),

$$\begin{aligned} |\beta(r)| & \leq \frac{\left| \int_{\mathbb{R}} \left\{ \alpha \text{sgn}(u_p) - \alpha \partial_1 \Psi(u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r, \cdot) \right\} \sin(k_0 x) dx \right|}{2(c^2 k_0 - 1)} \\ & \leq \frac{1}{2(c^2 k_0 - 1)} \int_{-6\epsilon/u'_p(0)}^{6\epsilon/u'_p(0)} \alpha \left( 1 + \frac{2}{\epsilon} |u_p + \xi(\beta)u_o + \gamma \sin(k_0 \cdot) - r| \right) k_0 |x| dx \\ & = O(1) \int_{-6\epsilon/u'_p(0)}^{6\epsilon/u'_p(0)} |x| dx = O(\epsilon^2). \end{aligned}$$

□

## 4 Two-transition solutions

In this section, we show the existence of travelling waves starting in one well of the on-site potential, making a transition to another well before returning to the first well. The on-site potential will be taken to be piecewise quadratic,  $\psi'(x) = \text{sgn}(x)$ , as in [11]. Also, we consider the same velocity regime  $c^2 \in [0.83, 1]$  as in that paper.

Our aim is to prove the existence of solutions representing two transitions between the two wells. We construct the solution similarly as in (11) for the case of a single transition, where the odd profile function  $u_p$  will be replaced by an even profile function  $v_p$ , and similarly the odd function  $u_o$  will be replaced by an even function  $u_e$ . That is, we use a decomposition of the form

$$u(x) = v_p(x) + \beta_e u_e(x) + \tilde{\gamma} \cos(k_0 x) - \tilde{r}(x). \quad (22)$$

Here  $v_p$  is the primary profile,  $\beta_e$  a small coefficient scaling the contribution from  $u_e$ ,  $\tilde{\gamma}$  a coefficient to be chosen later, and  $\tilde{r}$  a (small) remainder.

We first turn the attention to  $v_p$ .



**Lemma 4.1.** *Let  $x_0 \in (\pi/k_0)\mathbb{Z} = 2\mathbb{Z}$  be positive. Then there exist an even profile  $v_p \in H_{\text{loc}}^2(\mathbb{R})$  such that  $v_p$  vanishes exactly at the two points  $\pm x_0$ . Furthermore,*

$$\|(1+x^2)(Lv_p - \alpha \text{sgn}(v_p))\|_{L^2(\mathbb{R})} \rightarrow 0 \quad (23)$$

as  $x_0 \rightarrow \infty$ .

*Proof.* The odd solution  $x \rightarrow u_{\text{pa}}(x) - r(x)$  in [11] (see (7) and (8)) converges in  $H^2(z-2, z+2)$  as  $|z| \rightarrow \infty$  to the function

$$\text{sgn}(x) \left( A + B - B \cos(k_0 x) \right), \text{ where } A + B = 1 \text{ and } B = \frac{c^2 k_0^2 - 2}{c^2 k_0^2 - k_0},$$

where the expression for  $B$  makes use of (5) and (6).

It is straightforward to see that  $-u_{\text{pa}} + r$  is also a single-transition solution to the solution to the problem with piecewise quadratic on-site potential studied. We now introduce a two-transition profile  $v_p$  by combining these two single-transition solutions. Namely, for positive  $x_0 \in 2\mathbb{Z}$ , we define  $v_p$  as

$$v_p(x) := \left( \frac{1}{2} + \lambda(x) \right) (u_{\text{pa}}(x - x_0) - r(x - x_0)) \\ - \left( \frac{1}{2} - \lambda(x) \right) (u_{\text{pa}}(x + x_0) - r(x + x_0)),$$

where the step function  $\lambda \in C^\infty(\mathbb{R}, \mathbb{R})$  is odd and non-decreasing with  $\lambda(x) := -1/2$  for  $x \leq -1$  and  $\lambda(x) := 1/2$  for  $x \geq 1$ .

Obviously  $v_p$  is even, piecewise  $C^2$ , and satisfies  $Lv_p - \alpha \text{sgn}(v_p) = 0$  on  $\mathbb{R} \setminus [-2, 2]$ . To show (23), we thus only have to show that  $\|Lv_p - \alpha \text{sgn}(v_p)\|_{L^2(-2, 2)}$  tends to 0 as  $x_0 \rightarrow \infty$  with  $x_0 \in 2\mathbb{Z}$ . We first deal with  $x_0 \in 4\mathbb{Z}$ . For  $x \in (-2, 2)$ , we find that as  $x_0 \rightarrow \infty$

$$\begin{aligned} v_p(x) &\rightarrow \left( \frac{1}{2} + \lambda(x) \right) \text{sgn}(x - x_0) \{1 - B \cos(k_0(x - x_0))\} \\ &\quad - \left( \frac{1}{2} - \lambda(x) \right) \text{sgn}(x + x_0) \{1 - B \cos(k_0(x + x_0))\} \\ &= - \left( \frac{1}{2} + \lambda(x) \right) \{1 - B \cos(k_0(x - x_0))\} \\ &\quad - \left( \frac{1}{2} - \lambda(x) \right) \{1 - B \cos(k_0(x + x_0))\} \\ &= -1 + B \cdot \left\{ \left( \frac{1}{2} + \lambda(x) \right) \cos(k_0(x - x_0)) \right. \\ &\quad \left. + \left( \frac{1}{2} - \lambda(x) \right) \cos(k_0(x + x_0)) \right\} \\ &= -1 + B \cdot \{ \cos(k_0 x) \cos(k_0 x_0) + 2\lambda(x) \sin(k_0 x) \sin(k_0 x_0) \} \\ &= -1 + B \cdot \cos(k_0 x) \cos(k_0 x_0) =: v_p^\infty(x), \end{aligned}$$

as  $\sin(k_0 x_0) = 0$  and  $\cos(k_0 x_0) = 1$  is independent of  $x_0 \in 4\mathbb{Z}$ .

On  $(-2, 2)$ , this limit function  $v_p^\infty$  solves  $Lv_p^\infty - \alpha \operatorname{sgn}(v_p^\infty) = 0$ , since  $\cos(k_0 x_0) = 1$  and  $B = \frac{c^2 k_0^2 - 2}{c^2 k_0^2 - k_0} = 1 - \frac{2 - k_0}{c^2 k_0^2 - k_0} < 1$  gives

$$v_p^\infty(x) = -1 + B \cdot \cos(k_0 x) \cos(k_0 x_0) < 0$$

for all  $x \in (-2, 2)$ . Hence

$$Lv_p^\infty - \alpha \operatorname{sgn}(v_p^\infty) = B \cos(k_0 x_0) L \cos(k_0 \cdot) = 0.$$

As a consequence,  $\|Lv_p - \alpha \operatorname{sgn}(v_p)\|_{L^2(-2, 2)} \rightarrow 0$  as  $x_0 \in 4\mathbb{Z}$  tends to  $\infty$ .

The same argument works for  $x_0 \rightarrow \infty$  with  $x_0 \in 2\mathbb{Z} \setminus 4\mathbb{Z}$ , but this time  $\cos(k_0 x_0) = -1$ .  $\square$

Let us now turn to the even function  $u_e$ . For example, one can choose  $u_e$  to agree with  $\operatorname{sgn}(x) \sin(k_0 x)$  outside a fixed bounded interval. The essential property used is that such a function will satisfy the condition in Proposition A.2 in Appendix A.

For any choice of the parameter  $\beta_e \in \mathbb{R}$  and any  $\tilde{r} \in H_e^2(\mathbb{R})$ , we can choose the remaining parameter  $\tilde{\gamma}$  to ensure that  $u$  of (22) inherits the two zeros  $\pm x_0$  from  $v_p$ . That is, we set

$$\tilde{\gamma} := \{\tilde{r}(x_0) - \beta_e u_e(x_0)\} \cos(k_0 x_0)^{-1},$$

where we note that  $\cos(k_0 x_0) = \pm 1$  for  $x_0 \in 2\mathbb{Z}$ .

To motivate the definition of  $\tilde{r}$ , let us assume for the moment that  $\pm x_0$  are the only zeros of  $u$ . In other words, let us assume for now that the sign condition

$$\operatorname{sgn}(v_p + \beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}) = \operatorname{sgn}(v_p) \quad (24)$$

holds. In analogy to (13) as an equation for the remainder  $r$  in Section 3, we now consider the equation

$$L\tilde{r} = \beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p) \quad (25)$$

for  $\tilde{r} \in H_e^2(\mathbb{R})$ , where the subscript  $e$  stands for even functions. Note that if (25) has a solution  $\tilde{r}$ , then the function  $u$ , with the decomposition (22) will be a solution to (1) provided the sign condition (24) holds.

The solvability of (25) is addressed in the following lemma.

**Lemma 4.2.** *Define*

$$\beta_e := \frac{1}{2(c^2 k_0 - 1)} \int_{\mathbb{R}} [-L v_p + \alpha \operatorname{sgn}(v_p)] \cos(k_0 \cdot) dx.$$

*Then equation (25) has an even solution  $\tilde{r} \in H_e^2(\mathbb{R})$ . In particular, we have the estimate*

$$\|\tilde{r}\|_{H^2(\mathbb{R})} \leq C \left( |\beta_e| + \|(1 + x^2)(L v_p - \alpha \operatorname{sgn}(v_p))\|_{L^2(\mathbb{R})} \right).$$

*Proof.* By the choice of  $\beta_e$  and Proposition A.2,

$$\int_{\mathbb{R}} \left( \beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p) \right) \cos(k_0 \cdot) dx = 0.$$

The expression  $L^{-1}Q$  given by Proposition A.1 in Appendix A can be applied to the right hand side of (25),

$$Q := \beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p),$$

because  $(1 + x^2)Q \in L^2(\mathbb{R})$  and  $\int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = \int_{\mathbb{R}} Q(x) \cos(k_0 x) dx = 0$ . Hence

$$\tilde{r} := L^{-1}(\beta_e L u_e + L v_p - \alpha \operatorname{sgn}(v_p))$$

is well-defined. It is immediate that  $\tilde{r}$  is even.  $\square$

**Theorem 4.3.** *Under the assumptions of Theorem 2.1 (in particular, for a piecewise quadratic on-site potential,  $\psi'(x) = \operatorname{sgn}(x)$ ), there exists a family of even solutions*

$$u = v_p + \beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}$$

to (1), parametrised by the choice of sufficiently large  $x_0 \in 2\mathbb{Z}$  in Lemma 4.1.

Each of these solutions making two transitions between the wells of the on-site potential, located at  $-x_0$  and  $+x_0$ , so that they remain in one well only on a large but finite interval  $(-x_0, x_0)$ .

*Proof.* Lemma 4.1 provides  $v_p$ . Further,  $u_e$  is as discussed above. In addition, Lemma 4.2 defines  $\beta_e$  and  $\tilde{r}$ .

As

$$\beta_e = \frac{1}{2(c^2 k_0 - 1)} \int_{\mathbb{R}} [-L v_p + \alpha \operatorname{sgn}(v_p)] \cos(k_0 \cdot) dx,$$

we obtain by estimate (23)

$$|\beta_e| \leq C \left\| (1 + x^2) (L v_p - \alpha \operatorname{sgn}(v_p)) \right\|_{L^2(\mathbb{R})} \cdot \left\| \frac{\cos(k_0 x)}{1 + x^2} \right\|_{L^2(\mathbb{R})} \rightarrow 0$$

for a sequence of points  $x_0 \in 2\mathbb{Z}$  with  $x_0 \rightarrow \infty$ .

It remains to verify the sign condition (24) for  $u$ , i.e., to show that  $\pm x_0$  are the only roots of

$$u = v_p + \beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}.$$

Recall that the choice

$$\tilde{\gamma} = \{\tilde{r}(x_0) - \beta_e u_e(x_0)\} \cos(k_0 x_0)^{-1}$$

was made so that  $u$  vanishes at  $\pm x_0$ . The bounded embedding  $H^2(\mathbb{R}) \subset L^\infty(\mathbb{R})$  and Lemma 4.2 show that  $\tilde{r}(x_0)$  is small. Moreover, smallness of  $\beta_e$  and  $\tilde{r}(x_0)$  imply that  $\tilde{\gamma}$  is small itself.

As  $v_p$  changes sign at precisely  $\pm x_0$ , we now use that the derivative  $v_p'(\pm x_0)$  is bounded below independently of large  $x_0$ . Thus, pointwise smallness of all additional terms  $\beta_e u_e + \tilde{\gamma} \cos(k_0 \cdot) - \tilde{r}$  establishes the sign condition for all sufficiently large  $x_0$ .  $\square$

## A Appendix

We state a useful generalisation of Proposition 2.3, by considering functions  $Q$  which are not necessarily odd.

**Proposition A.1.** *If  $Q \in L^2(\mathbb{R})$  satisfies*

$$(1+x^2)Q \in L^2(\mathbb{R}) \quad \text{and} \quad \int_{\mathbb{R}} Q(x) \sin(k_0 x) dx = \int_{\mathbb{R}} Q(x) \cos(k_0 x) dx = 0,$$

*then, for all  $c$  near enough to 1, there exists a unique function  $r \in H^2(\mathbb{R})$  such that  $Lr = c^2 r'' - \Delta_D r + \alpha r = Q$ . Moreover,*

$$\|r\|_{H^2(\mathbb{R})} := \|(1+k^2)\hat{r}\|_{L^2(\mathbb{R})} \leq (C_1 + ((4+\alpha)C_1 + 1)/c^2) \|(1+x^2)Q\|_{L^2(\mathbb{R})},$$

*where the constant  $C_1 > 0$  is as in Proposition 2.3.*

*Proof.* When  $Q$  is even, the proof is the same as the one of Proposition 2.3, except that then  $\hat{Q}$ ,  $\hat{r}$  and  $r$  are even and real-valued. In general, we write  $Q = Q_o + Q_e$ , where

$$Q_o(x) = \frac{1}{2}(Q(x) - Q(-x)) \quad \text{and} \quad Q_e(x) = \frac{1}{2}(Q(x) + Q(-x))$$

are odd respectively even. We set

$$\hat{r}_o(k) := \frac{\hat{Q}_o(k)}{D(k)} \quad \text{and} \quad \hat{r}_e(k) := \frac{\hat{Q}_e(k)}{D(k)}, \quad k \in \mathbb{R},$$

which are odd respectively even as well. Then  $r := r_o + r_e$  satisfies

$$\hat{r}(k) = \frac{\hat{Q}(k)}{D(k)}, \quad k \in \mathbb{R}.$$

As

$$\int_{\mathbb{R}} (1+k^2)^2 \hat{r}_o(k) \cdot \overline{\hat{r}_e(k)} dk = \int_{\mathbb{R}} (1+x^2)^2 Q_o(x) Q_e(x) dx = 0,$$

we obtain

$$\begin{aligned} \|r\|_{H^2(\mathbb{R})}^2 &:= \|(1+k^2)\hat{r}\|_{L^2(\mathbb{R})}^2 = \|(1+k^2)\hat{r}_o\|_{L^2(\mathbb{R})}^2 + \|(1+k^2)\hat{r}_e\|_{L^2(\mathbb{R})}^2 \\ &\leq (C_1 + ((4+\alpha)C_1 + 1)/c^2)^2 \left( \|(1+x^2)Q_o\|_{L^2(\mathbb{R})}^2 + \|(1+x^2)Q_e\|_{L^2(\mathbb{R})}^2 \right) \\ &= (C_1 + ((4+\alpha)C_1 + 1)/c^2)^2 \|(1+x^2)Q\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

□

The following proposition establishes orthogonality relations and estimates for the Fourier mode associated with  $k_0$  for  $L$  applied to even and odd functions. The estimate (26) is used just after the compactness proof (Lemma 3.3).

**Proposition A.2.** Consider the odd function  $u_o \in H_{\text{loc}}^2(\mathbb{R})$  satisfying (12). In addition, let  $u_e \in H_{\text{loc}}^2(\mathbb{R})$  be an even function such that

$$(1+x^2) \frac{d^l}{dx^l} (u_e(x) - \text{sgn}(x) \sin(k_0 x)) \in L^2(\mathbb{R} \setminus [-1, 1])$$

for  $l = 0, 1, 2$ , analogously to (12). If  $c > k_0^{-1/2}$ , then

$$\begin{aligned} \int_{\mathbb{R}} \sin(k_0 \cdot) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) dx &= -2c^2 k_0 + 2 < 0, \\ \int_{\mathbb{R}} \cos(k_0 \cdot) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx &= 2c^2 k_0 - 2 > 0, \\ \int_{\mathbb{R}} \cos(k_0 \cdot) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) dx &= 0 \end{aligned} \quad (26)$$

and

$$\int_{\mathbb{R}} \sin(k_0 \cdot) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx = 0.$$

*Proof.* The two last integrals vanish because the integrands are odd functions of  $x$ . For the first integral, two integrations by parts and the identity  $L \sin(k_0 \cdot) = 0$  give

$$\begin{aligned} & \lim_{z \rightarrow \infty} \int_{-z}^z \sin(k_0 \cdot) (c^2 u_o'' - \Delta_D u_o + \alpha u_o) dx \\ &= \lim_{z \rightarrow \infty} \int_{-z}^z \left[ c^2 \frac{d^2}{dx^2} \sin(k_0 \cdot) - \Delta_D \sin(k_0 \cdot) + \alpha \sin(k_0 \cdot) \right] u_o dx \\ & \quad + \lim_{z \rightarrow \infty} c^2 [\sin(k_0 z) u_o'(z) - k_0 \cos(k_0 z) u_o(z) \\ & \quad - \sin(-k_0 z) u_o'(-z) + k_0 \cos(-k_0 z) u_o(-z)] \\ & \quad - \lim_{z \rightarrow \infty} \left( \int_{-z+1}^{z+1} - \int_{-z}^z \right) \sin(k_0(x-1)) u_o(x) dx \\ & \quad - \lim_{z \rightarrow \infty} \left( \int_{-z-1}^{z-1} - \int_{-z}^z \right) \sin(k_0(x+1)) u_o(x) dx \\ & \stackrel{(12)}{=} \lim_{z \rightarrow \infty} c^2 (-k_0 \sin^2(k_0 z) - k_0 \cos^2(k_0 z) - k_0 \sin^2(-k_0 z) - k_0 \cos^2(-k_0 z)) \\ & \quad - \lim_{z \rightarrow \infty} \int_z^{z+1} \sin(k_0(x-1)) \cos(k_0 x) dx - \lim_{z \rightarrow \infty} \int_{-z}^{-z+1} \sin(k_0(x-1)) \cos(k_0 x) dx \\ & \quad + \lim_{z \rightarrow \infty} \int_{-z-1}^{-z} \sin(k_0(x+1)) \cos(k_0 x) dx + \lim_{z \rightarrow \infty} \int_{z-1}^z \sin(k_0(x+1)) \cos(k_0 x) dx \\ &= -2c^2 k_0 + \lim_{z \rightarrow \infty} \int_{z-1}^{z+1} \cos^2(k_0 x) dx + \lim_{z \rightarrow \infty} \int_{-z-1}^{-z+1} \cos^2(k_0 x) dx \\ &= -2c^2 k_0 + 2 < 0. \end{aligned}$$

Analogously,

$$\begin{aligned}
& \int_{\mathbb{R}} \cos(k_0 \cdot) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx = \int_{\mathbb{R}} \sin(k_0 \cdot + k_0) (c^2 u_e'' - \Delta_D u_e + \alpha u_e) dx \\
& = \lim_{z \rightarrow \infty} c^2 (-k_0 \sin(k_0 z + k_0) \sin(k_0 z - k_0) - k_0 \cos(k_0 z + k_0) \cos(k_0 z - k_0) \\
& \quad - k_0 \sin(-k_0 z + k_0) \sin(-k_0 z - k_0) - k_0 \cos(-k_0 z + k_0) \cos(-k_0 z - k_0)) \\
& \quad - \lim_{z \rightarrow \infty} \int_z^{z+1} \sin(k_0(x-1) + k_0) \cos(k_0 x - k_0) dx \\
& \quad - \lim_{z \rightarrow \infty} \int_{-z}^{-z+1} \sin(k_0(x-1) + k_0) \cos(k_0 x - k_0) dx \\
& \quad + \lim_{z \rightarrow \infty} \int_{-z-1}^{-z} \sin(k_0(x+1) + k_0) \cos(k_0 x - k_0) dx \\
& \quad + \lim_{z \rightarrow \infty} \int_{z-1}^z \sin(k_0(x+1) + k_0) \cos(k_0 x - k_0) dx \\
& = 2c^2 k_0 - 2 > 0.
\end{aligned}$$

□

**Acknowledgement** This work was initiated at the workshop “Solitons, Vortices, Minimal Surfaces and their Dynamics” at the Mittag Leffler Institute in 2013. JZ greatly acknowledges funding by the EPSRC, EP/K027743/1.

## References

- [1] Sigurd Angenent. The shadowing lemma for elliptic PDE. In *Dynamics of infinite-dimensional systems (Lisbon, 1986)*, volume 37 of *NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci.*, pages 7–22. Springer, Berlin, 1987.
- [2] W. Atkinson and N. Cabrera. Motion of a Frenkel-Kontorowa dislocation in a one-dimensional crystal. *Phys. Rev.*, 138(3A):A763–A766, May 1965.
- [3] Oleg M. Braun and Yuri S. Kivshar. Nonlinear dynamics of the Frenkel-Kontorova model. *Phys. Rep.*, 306(1-2):108, 1998.
- [4] Renato Calleja and Yannick Sire. Travelling waves in discrete nonlinear systems with non-nearest neighbour interactions. *Nonlinearity*, 22(11):2583–2605, 2009.
- [5] Y. Y. Earmme and J. H. Weiner. Breakdown phenomena in high-speed dislocations. *J. Appl. Phys.*, 45(2):603–609, 1974.
- [6] J. Frenkel and T. Kontorova. On the theory of plastic deformation and twinning. *Acad. Sci. U.S.S.R. J. Phys.*, 1:137–149, 1939.

- [7] Gero Friesecke and Jonathan A. D. Wattis. Existence theorem for solitary waves on lattices. *Comm. Math. Phys.*, 161(2):391–418, 1994.
- [8] Jack Hale. *Theory of functional differential equations*. Springer-Verlag, New York-Heidelberg, second edition, 1977. Applied Mathematical Sciences, Vol. 3.
- [9] Michael Herrmann, Karsten Matthies, Hartmut Schwetlick, and Johannes Zimmer. Subsonic phase transition waves in bistable lattice models with small spinodal region. *SIAM J. Math. Anal.*, 45(5):2625–2645, 2013.
- [10] Gérard Iooss and Klaus Kirchgässner. Travelling waves in a chain of coupled nonlinear oscillators. *Comm. Math. Phys.*, 211(2):439–464, 2000.
- [11] Carl-Friedrich Kreiner and Johannes Zimmer. Existence of subsonic heteroclinic waves for the Frenkel-Kontorova model with piecewise quadratic on-site potential. *Nonlinearity*, 24(4):1137–1163, 2011.
- [12] O. Kresse and L. Truskinovsky. Mobility of lattice defects: discrete and continuum approaches. *J. Mech. Phys. Solids*, 51(7):1305–1332, 2003.
- [13] R. S. MacKay and S. Aubry. Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators. *Nonlinearity*, 7(6):1623–1643, 1994.
- [14] Michel Peyrard and Martin D. Kruskal. Kink dynamics in the highly discrete sine-Gordon system. *Phys. D*, 14(1):88–102, 1984.
- [15] Hartmut Schwetlick and Johannes Zimmer. Existence of dynamic phase transitions in a one-dimensional lattice model with piecewise quadratic interaction potential. *SIAM J. Math. Anal.*, 41(3):1231–1271, 2009.